Nonlinear Wave Modulation in a Fluid-Filled Elastic Tube with Stenosis

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In the present work, treating the arteries as a thin-walled and prestressed elastic tube with a stenosis and the blood as a Newtonian fluid with negligible viscosity, we have studied the amplitude modulation of nonlinear waves in such a composite system by use of the reductive perturbation method. The governing evolution equation was obtained as the variable coefficient nonlinear Schrödinger (NLS) equation. By setting the stenosis function equal to zero, we observed that this variable coefficient NLS equation reduces to the conventional NLS equation. After introducing a new dependent variable and a set of new independent coordinates, we reduced the evolution equation to the conventional NLS equation. By seeking a progressive wave type of solution to this evolution equation we observed, that the wave trajectories are not straight lines anymore; they are rather some curves in the (ξ, τ) plane. It was further observed that the wave speeds for both enveloping and harmonic waves are variable, and the speed of the enveloping wave increases with increasing axial distance, whereas the speed of the harmonic wave decreases with increasing axial coordinates. The numerical calculations indicated that the speed of the harmonic wave decreases with increasing time parameter, but the sensitivity of wave speed to this parameter is quite weak.

Key words: Solitary Waves; Tubes with Stenosis; Wave Modulation.

1. Introduction

The striking feature of the arterial blood flow is its pulsatile character. The intermittent ejection of blood from the left ventricle produces pressure and flow pulses in the arterial tree. Experimental studies revealed that the flow velocity in blood vessels largely depends on the elastic properties of the vessel wall, and they propagate towards the periphery with a characteristic pattern [1].

Due to its applications in arterial mechanics, the propagation of pressure pulses in fluid-filled distensible tubes has been studied by several researchers (Pedley [2] and Fung [3]). Most of the works on wave propagation in compliant tubes have considered small amplitude waves ignoring the nonlinear effects and focusing on the dispersive character of waves (see Atabek and Lew [4], Rachev [5] and Demiray [6]). However, when the nonlinear terms arising from the constitutive equations and/or kinematical relations are introduced, one has to consider either finite amplitude or small-but-finite amplitude waves, depending on the order of nonlinearity.

The propagation of finite amplitude waves in fluid-filled elastic or viscoelastic tubes has been examined, for instance, by Rudinger [7], Anliker et al. [8] and Tait and Moodie [9] by using the method of characteristics in studying the shock formation. On the other hand, the propagation of small-but-finite amplitude waves in distensible tubes has been investigated by Johnson [10], Hashizume [11], Yomosa [12] and Demiray [13] by employing various asymptotic methods. In all these works, depending on the balance between nonlinearity, dispersion and dissipation, the Korteweg – de Vries (KdV), Burger's or KdV – Burger's equations are obtained as the evolution equations.

As is well known, when the nonlinear effects are small, the system of equations that describe the physical phenomenon admit a harmonic wave solution with constant amplitude. If the amplitude of the wave is small but finite, the nonlinear terms cannot be neglected, and the nonlinearity gives rise to the variation of amplitude both in space and time variables. Then the amplitude varies slowly over a period of oscillations. A stretching transformation allows to decompose the system into a rapidly varying part associated with the oscillation and a slowly varying part such as the amplitude. A formal solution can be given in the form of an asymptotic expansion, and an equation determining the modulation of the first-order amplitude can be derived. For instance, the nonlinear Schrödinger (NLS) equation is the simplest representative equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media. It exhibits a balance between nonlinearity and dispersion. The problem of self-modulation of small-but-finite amplitude waves in fluid-filled distensible tubes was considered by Ravindran and Prasad [14], who showed that, for a linear elastic tube wall model, the nonlinear self-modulation of pressure waves is governed by the NLS equation. Demiray [15], employing the exact equations of a viscous fluid and of a prestressed elastic tube, studied the amplitude modulation of nonlinear waves and obtained the dissipative NLS equation as the governing equation. In all these works, the arteries are considered as cylindrical tubes with constant radius. In essence, the radius of the arteries is variable along the axis of the tube.

In the present work, treating the arteries as a thin-walled and prestressed elastic tube with a stenosis and the blood as a Newtonian fluid with negligible viscosity, we have studied the amplitude modulation of nonlinear waves in such a composite system by use of the reductive perturbation method. The governing evolution equation was obtained as the variable coefficient nonlinear Schrödinger equation. By introducing a new dependent variable and a set of new independent coordinates, we reduced the evolution equation to the conventional NLS equation. By seeking a progressive wave type of solution to this evolution equation we observed, that the speed of the enveloping wave increases with increasing radius, whereas the speed of the harmonic wave decreases with increasing radius. We further noticed that the speed of the harmonic wave decreases with increasing time parameter, but the sensitivity of wave speed to this parameter is quite low.

2. Basic Equations and Theoretical Preliminaries

2.1. Equations of the Tube

In this section, we shall derive the basic equations governing the motion of a prestressed thin elastic tube, with an axially symmetric bump (stenosis), and filled with a nonviscous fluid. For that purpose, we consider a circularly cylindrical tube of radius R_0. It is assumed that such a tube is subjected to an axial stretch λ_z and a static inner pressure $P_0^*(Z^*)$. Under the effect of such a variable pressure, the position vector of a generic point on the tube is assumed to be described by

$$\boldsymbol{r}_0 = [r_0 - f^*(z^*)]\boldsymbol{e_r} + z^*\boldsymbol{e_z}, \quad z^* = \lambda_z Z^*, \tag{1}$$

where $\boldsymbol{e_r}$, $\boldsymbol{e_\theta}$ and $\boldsymbol{e_z}$ are the unit base vectors in the cylindrical polar coordinates, r_0 is the deformed radius at the origin of the coordinate system, Z^* the axial coordinate before the deformation, z^* the axial coordinate after static deformation, and $f^*(z^*)$ a function that characterizes the axially symmetric stenosis on the surface of the arterial wall and will be specified later.

Upon this initial static deformation, we shall superimpose a dynamical radial displacement $u^*(z^*,t^*)$, where t^* is the time parameter, but, in view of the external tethering, the axial displacement is assumed to be negligible. Then the position vector $\boldsymbol{r}$ of a generic point on the tube may be described by

$$\boldsymbol{r} = [r_0 - f^*(z^*) + u^*]\boldsymbol{e_r} + z^*\boldsymbol{e_z}. \tag{2}$$

The arc lengths along the meridional and circumferential curves are, respectively, given by

$$\begin{aligned} \mathrm{d}s_z &= \left[1 + \left(-f^{*'} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} \mathrm{d}z^*, \\ \mathrm{d}s_\theta &= (r_0 - f^* + u^*)\mathrm{d}\theta. \end{aligned} \tag{3}$$

Then the stretch ratios along the meridional and circumferential curves may, respectively, be given by

$$\begin{aligned} \lambda_1 &= \lambda_z \left[1 + \left(-f^{*'} + \partial u^*/\partial z^*\right)^2\right]^{1/2}, \\ \lambda_2 &= \frac{1}{R_0}(r_0 - f^* + u^*), \end{aligned} \tag{4}$$

where the prime denotes the differentiation of the corresponding quantity with respect to z^*. The unit tangent vector $\boldsymbol{t}$ along the deformed meridional curve and the unit exterior normal vector $\boldsymbol{n}$ to the deformed tube are given by

$$\begin{aligned} \boldsymbol{t} &= \frac{(-f^{*'} + \partial u^*/\partial z^*)\boldsymbol{e_r} + \boldsymbol{e_z}}{[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}}, \\ \boldsymbol{n} &= \frac{\boldsymbol{e_r} - (-f^{*'} + \partial u^*/\partial z^*)\boldsymbol{e_z}}{[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}}. \end{aligned} \tag{5}$$

Let T_1 and T_2 be the membrane forces along the meridional and circumferential curves, respectively. Then the equation of the radial motion of a small tube element placed between the planes z^* = const,

$z^* + \mathrm{d}z^* = \text{const}$, $\theta = \text{const}$ and $\theta + \mathrm{d}\theta = \text{const}$ may be given by

$$-T_2\left[1+\left(-f^{*'}+\frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2}$$
$$+\frac{\partial}{\partial z^*}\left\{\frac{(r_0-f^*+u^*)(-f^{*'}+\partial u^*/\partial z^*)}{[1+(f^{*'}+\partial u^*/\partial z^*)^2]^{1/2}}T_1\right\} \quad (6)$$
$$+P^*(r_0-f^*+u^*)=\rho_0\frac{HR_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}},$$

where ρ_0 is the mass density of the tube, H the thickness in the undeformed configuration and P^* the inner pressure applied by the fluid.

Let $\mu\Sigma$ be the strain energy density function of the membrane, where μ is the shear modulus of the tube material. Then the membrane forces may be expressed in terms of the stretch ratios as

$$T_1=\frac{\mu H}{\lambda_2}\frac{\partial\Sigma}{\partial\lambda_1},\quad T_2=\frac{\mu H}{\lambda_1}\frac{\partial\Sigma}{\partial\lambda_2}. \quad (7)$$

Introducing (7) into (6), the equation of motion of the tube in the radial direction takes the form

$$-\frac{\mu}{\lambda_z}\frac{\partial\Sigma}{\partial\lambda_2}$$
$$+\mu R_0\frac{\partial}{\partial z^*}\left\{\frac{(-f^{*'}+\partial u^*/\partial z^*)}{[1+(-f^{*'}+\partial u^*/\partial z^*)^2]^{1/2}}\frac{\partial\Sigma}{\partial\lambda_1}\right\} \quad (8)$$
$$+\frac{P^*}{H}(r_0-f^*+u^*)=\rho_0\frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}.$$

2.2. Equations of the Fluid

In general, blood is known to be an incompressible non-Newtonian fluid. The main factor for blood to behave like a non-Newtonian fluid is the level of cell concentration (hematocrit ratio) and the deformability of red blood cells. In the course of blood flow in arteries, the red cells move to the central region of the artery and, thus, the hematocrit ratio is reduced near the arterial wall, where the shear rate is quite high, as can be seen from Poiseuille flow. Experimental studies indicate that, when the hematocrit ratio is low and the shear rate is high, blood behaves like a Newtonian fluid [2]. Moreover, as pointed out by Rudinger [7], for flows in large blood vessels, the viscosity of blood may be neglected as a first approximation. For this case, the variation of the field quantities with the radial coordinate may be neglected. Thus, the averaged equations of motion of an incompressible fluid may be given by

$$\frac{\partial A^*}{\partial t^*}+\frac{\partial}{\partial z^*}(A^*w^*)=0, \quad (9)$$

$$\frac{\partial w^*}{\partial t^*}+w^*\frac{\partial w^*}{\partial z^*}+\frac{1}{\rho_\mathrm{f}}\frac{\partial P^*}{\partial z^*}=0, \quad (10)$$

where A^* is the cross-sectional area of the tube, w^* the averaged axial fluid velocity, and P^* the averaged fluid pressure. Noting the relation between the cross-sectional area and the final radius, i.e., $A^*=\pi(r_0-f^*+u^*)^2$, (9) reads

$$2\frac{\partial u^*}{\partial t^*}+2w^*\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)+(r_0-f^*+u^*)\frac{\partial w^*}{\partial z^*}=0. \quad (11)$$

At this stage it is convenient to introduce the following nondimensionalized quantities:

$$t^*=\left(\frac{R_0}{c_0}\right)t,\quad z^*=R_0 z,\quad u^*=R_0 u,$$
$$m=\frac{\rho_0 H}{\rho_\mathrm{f}R_0},\quad w^*=c_0 w,\quad f^*=R_0 f, \quad (12)$$
$$r_0=R_0\lambda_\theta,\quad P^*=\rho_\mathrm{f}c_0^2 p,\quad c_0^2=\frac{\mu H}{\rho_\mathrm{f}R_0}.$$

Introducing (12) into equations (8), (9) and (10) we obtain the nondimensionalized equations

$$2\frac{\partial u}{\partial t}+2\left(-f'+\frac{\partial u}{\partial z}\right)w+(\lambda_\theta-f+u)\frac{\partial w}{\partial z}=0, \quad (13)$$

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}+\frac{\partial p}{\partial z}=0, \quad (14)$$

$$p=\frac{m}{\lambda_z(\lambda_\theta-f+u)}\frac{\partial^2 u}{\partial t^2}+\frac{1}{\lambda_z(\lambda_\theta-f+u)}\frac{\partial\Sigma}{\partial\lambda_2}$$
$$-\frac{1}{(\lambda_\theta-f+u)}\frac{\partial}{\partial z}\left\{\frac{(-f'+\partial u/\partial z)}{[1+(-f'+\partial u/\partial z)^2]^{1/2}}\frac{\partial\Sigma}{\partial\lambda_1}\right\}. \quad (15)$$

For our future purposes we need the power series expansion of p in terms of $\bar{u}=u-f$ and its temporal and spatial derivatives. If the expression (15) is expanded into a power series in u, up to and including the cubic terms, we obtain

$$p=p_0+L_1(u)+L_2(u)+L_3(u), \quad (16)$$

where

$$p_0 = \frac{1}{\lambda_\theta \lambda_z} \frac{\partial \Sigma}{\partial \lambda_\theta},$$

$$L_1(u) = \frac{m}{\lambda_\theta \lambda_z} \frac{\partial^2 \bar{u}}{\partial t^2} + \beta_1 \bar{u} - \alpha_0 \frac{\partial^2 \bar{u}}{\partial z^2},$$

$$L_2(u) = -\frac{m}{\lambda_\theta^2 \lambda_z} \bar{u} \frac{\partial^2 \bar{u}}{\partial t^2} + \beta_2 \bar{u}^2 - \alpha_1 \bar{u} \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{1}{2}\left(\alpha_1 + \frac{\alpha_0}{\lambda_\theta}\right)\left(\frac{\partial \bar{u}}{\partial z}\right)^2, \tag{17}$$

$$L_3(u) = \frac{m}{\lambda_\theta^3 \lambda_z} \bar{u}^2 \frac{\partial^2 \bar{u}}{\partial t^2} + \beta_3 \bar{u}^3 - \alpha_2 \bar{u}^2 \frac{\partial \bar{u}}{\partial z^2} - \left(\alpha_2 + \frac{\alpha_1}{2\lambda_\theta} - \frac{\alpha_0}{2\lambda_\theta^2}\right) \bar{u} \left(\frac{\partial \bar{u}}{\partial z}\right)^2 - \frac{3}{2}(\gamma_1 - \alpha_0)\left(\frac{\partial \bar{u}}{\partial z}\right)^2 \frac{\partial^2 \bar{u}}{\partial z^2}.$$

Here, the coefficients α_0, α_1, α_2, β_1, β_2, β_3, and γ_1 are defined by

$$\alpha_0 = \frac{1}{\lambda_\theta} \frac{\partial \Sigma}{\partial \lambda_z}, \quad \alpha_1 = \frac{1}{\lambda_\theta} \frac{\partial^2 \Sigma}{\partial \lambda_\theta \lambda_z}, \quad \alpha_2 = \frac{1}{2\lambda_\theta} \frac{\partial^3 \Sigma}{\partial \lambda_\theta^2 \partial \lambda_z},$$

$$\beta_1 = \frac{1}{\lambda_\theta \lambda_z}\left(\frac{\partial^2 \Sigma}{\partial \lambda_\theta^2} - \frac{1}{\lambda_\theta} \frac{\partial \Sigma}{\partial \lambda_\theta}\right),$$

$$\beta_2 = \frac{1}{2\lambda_\theta \lambda_z} \frac{\partial^3 \Sigma}{\partial \lambda_\theta^3} - \frac{\beta_1}{\lambda_\theta}, \quad \beta_3 = \frac{1}{6\lambda_\theta \lambda_z} \frac{\partial^4 \Sigma}{\partial \lambda_\theta^4} - \frac{\beta_2}{\lambda_\theta},$$

$$\gamma_1 = \frac{\lambda_z}{\lambda_\theta} \frac{\partial^2 \Sigma}{\partial \lambda_z^2}. \tag{18}$$

These equations give sufficient relations to determine the field variables u, w and p completely.

3. Nonlinear Wave Modulation

In this section, we will examine the amplitude modulation of weakly nonlinear waves in a fluid-filled elastic tube with a stenosis, whose dimensionless governing equations are given in (13), (14) and (16). Considering the dispersion relation of the linearized field equations and the nature of the problem of concern, which is a boundary value problem, the following stretched coordinates may be introduced:

$$\xi = \varepsilon(z - \lambda t), \quad \tau = \varepsilon^2 z, \tag{19}$$

where ε is a small parameter measuring the weakness of nonlinearity, and λ is a constant to be determined from the solution. Solving z in terms of the variable τ, we get $z = \tau/\varepsilon^2$. Introducing this expression of z into the expression of $f(z)$, we obtain

$$f(z) = f(\tau/\varepsilon^2) = \hat{h}(\varepsilon, \tau). \tag{20}$$

Here, we shall assume that $\hat{h}(\varepsilon, \tau)$ is of the form $\hat{h}(\varepsilon, \tau) = \varepsilon h(\tau)$ and the field variables u, w and p are functions of the slow variables (ξ, τ) as well as the fast variables (z, t). We further assume that the field variables may be expanded into asymptotic series as

$$\begin{aligned} u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots, \\ w &= \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \dots, \\ p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \dots. \end{aligned} \tag{21}$$

Noting the differential relations

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon \lambda \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} \to \frac{\partial}{\partial z} + \varepsilon \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \tag{22}$$

introducing the expansions (21) and (22) into the field equations (13), (14) and (16), and equating the coefficients of like powers of ε equal to zero, we obtain the following sets of differential equations.

$O(\varepsilon)$ *equations*:

$$\begin{aligned} &2\frac{\partial u_1}{\partial t} + \lambda_\theta \frac{\partial w_1}{\partial z} = 0, \quad \frac{\partial w_1}{\partial t} + \frac{\partial p_1}{\partial z} = 0, \\ &p_1 = \frac{m}{\lambda_\theta \lambda_z} \frac{\partial^2 u_1}{\partial t^2} - \alpha_0 \frac{\partial^2 u_1}{\partial z^2} + \beta_1 (u_1 - h). \end{aligned} \tag{23}$$

$O(\varepsilon^2)$ *equations*:

$$2\frac{\partial u_2}{\partial t} + \lambda_\theta \frac{\partial w_2}{\partial z} - 2\lambda \frac{\partial u_1}{\partial \xi} + \lambda_\theta \frac{\partial w_1}{\partial \xi} + (u_1 - h)\frac{\partial w_1}{\partial z} = 0,$$

$$\frac{\partial w_2}{\partial t} + \frac{\partial p_2}{\partial z} - \lambda \frac{\partial w_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} + w_1 \frac{\partial w_1}{\partial z} = 0,$$

$$p_2 = \frac{m}{\lambda_\theta \lambda_z} \frac{\partial^2 u_2}{\partial t^2} - \alpha_0 \frac{\partial^2 u_2}{\partial z^2} + \beta_1 u_2 - \frac{2m\lambda}{\lambda_\theta \lambda_z} \frac{\partial^2 u_1}{\partial \xi \partial t} - 2\alpha_0 \frac{\partial^2 u_1}{\partial \xi \partial z} - \frac{m}{\lambda_\theta^2 \lambda_z}(u_1 - h)\frac{\partial^2 u_1}{\partial t^2} - \frac{1}{2}\left(\alpha_1 + \frac{\alpha_0}{\lambda_\theta}\right)\left(\frac{\partial u_1}{\partial z}\right)^2 - \alpha_1 (u_1 - h)\frac{\partial^2 u_1}{\partial z^2} + \beta_2 (u_1 - h)^2. \tag{24}$$

$O(\varepsilon^3)$ *equations*:

$$2\frac{\partial u_3}{\partial t}+\lambda_\theta\frac{\partial w_3}{\partial z}-2\lambda\frac{\partial u_2}{\partial\xi}+\lambda_\theta\frac{\partial w_2}{\partial\xi}+w_1\left(\frac{\partial u_1}{\partial\xi}+\frac{\partial u_2}{\partial z}\right)+w_2\frac{\partial u_1}{\partial z}+\lambda_\theta\frac{\partial w_1}{\partial\tau}+(u_1-h)\left(\frac{\partial w_2}{\partial z}+\frac{\partial w_1}{\partial\xi}\right)+u_2\frac{\partial w_1}{\partial z}=0,$$

$$\frac{\partial w_3}{\partial t}+\frac{\partial p_3}{\partial z}-\lambda\frac{\partial w_2}{\partial\xi}+\frac{\partial p_2}{\partial\xi}+\frac{\partial p_1}{\partial\tau}+\frac{\partial}{\partial z}(w_1w_2)+w_1\frac{\partial w_1}{\partial\xi}=0,$$

$$\begin{aligned}p_3={}&\frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2u_3}{\partial t^2}-\alpha_0\frac{\partial^2u_3}{\partial z^2}+\beta_1u_3-\frac{2m\lambda}{\lambda_\theta\lambda_z}\frac{\partial^2u_2}{\partial\xi\partial t}\\&-2\alpha_0\frac{\partial^2u_2}{\partial\xi\partial z}-\alpha_0\left(\frac{\partial^2u_1}{\partial\xi^2}+2\frac{\partial^2u_1}{\partial z\partial\tau}\right)+\frac{m\lambda^2}{\lambda_\theta\lambda_z}\frac{\partial^2u_1}{\partial\xi^2}\\&-\frac{m}{\lambda_\theta^2\lambda_z}(u_1-h)\left(\frac{\partial^2u_2}{\partial t^2}-2\lambda\frac{\partial^2u_1}{\partial\xi\partial t}\right)\\&-\frac{m}{\lambda_\theta^2\lambda_z}u_2\frac{\partial^2u_1}{\partial t^2}-\left(\alpha_1+\frac{\alpha_0}{\lambda_\theta}\right)\frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial z}+\frac{\partial u_1}{\partial\xi}\right)\\&-\alpha_1(u_1-h)\left(\frac{\partial^2u_2}{\partial z^2}+2\frac{\partial^2u_1}{\partial z\partial\xi}\right)-\alpha_1u_2\frac{\partial^2u_1}{\partial z^2}\\&+2\beta_2(u_1-h)u_2+\frac{m}{\lambda_\theta^3\lambda_z}(u_1-h)^2\frac{\partial^2u_1}{\partial t^2}\\&-\left(\alpha_2+\frac{\alpha_1}{2\lambda_\theta}-\frac{\alpha_0}{2\lambda_\theta^2}\right)(u_1-h)\left(\frac{\partial u_1}{\partial z}\right)^2\\&-\alpha_2(u_1-h)^2\frac{\partial^2u_1}{\partial z^2}\\&-\frac{3}{2}(\gamma_1-\alpha_0)\left(\frac{\partial u_1}{\partial z}\right)^2\frac{\partial^2u_1}{\partial z^2}+\beta_3(u_1-h)^3.\end{aligned}\quad(25)$$

4. Solution of the Field Equations

4.1. Solution of $O(\varepsilon)$ Equations

The form of the differential equations (23) suggests to seek the following type of solution to these differential equations:

$$\begin{aligned}u_1&=(U_1\mathrm{e}^{\mathrm{i}\theta}+c.c.),\\w_1&=\frac{2}{\lambda_\theta}\frac{\omega}{k}w_0h(\tau)+(W_1\mathrm{e}^{\mathrm{i}\theta}+c.c.),\\p_1&=-\beta_1h(\tau)+\left[-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right]U_1\mathrm{e}^{\mathrm{i}\theta}+c.c.,\end{aligned}\quad(26)$$

where ω is the angular frequency, k the wave number, $(2\omega/\lambda_\theta k)w_0h(\tau)$ the steady flow corresponding to the steady pressure $-\beta_1h(\tau)$, w_0 a constant to be determined from the solution; U_1 and W_1 are unknown functions of the slow variables (ξ,τ); $\theta=\omega t-kz$ is the phasor and $c.c.$ stands for the complex conjugate of the corresponding expressions. Introducing (26) into (23) we have

$$U_1=U(\xi,\tau),\quad W_1=\frac{2}{\lambda_\theta}\frac{\omega}{k}U,\quad(27)$$

provided that the following dispersion relation holds true:

$$\omega^2=\frac{\lambda_\theta(\alpha_0k^2+\beta_1)k^2}{2+mk^2/\lambda_z}.\quad(28)$$

Here $U(\xi,\tau)$ is a unknown function whose governing equation will be obtained later.

4.2. Solution of $O(\varepsilon^2)$ Equations

Introducing the solution given in (27) into equations (24) we have

$$\begin{aligned}&2\frac{\partial u_2}{\partial t}+\lambda_\theta\frac{\partial w_2}{\partial z}+\left[2\left(\frac{\omega}{k}-\lambda\right)\frac{\partial U}{\partial\xi}+2\mathrm{i}\frac{\omega}{\lambda_\theta}hU\right]\mathrm{e}^{\mathrm{i}\theta}-2\mathrm{i}\frac{\omega}{\lambda_\theta}U^2\mathrm{e}^{2\mathrm{i}\theta}+c.c.=0,\\&\frac{\partial w_2}{\partial t}+\frac{\partial p_2}{\partial z}+\left[\left(-2\frac{\lambda}{\lambda_\theta}\frac{\omega}{k}-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right)\frac{\partial U}{\partial\xi}-4\mathrm{i}\frac{\omega^2}{\lambda_\theta^2k}w_0h(\tau)U\right]\mathrm{e}^{\mathrm{i}\theta}-4\mathrm{i}\frac{\omega^2}{\lambda_\theta^2k}U^2\mathrm{e}^{2\mathrm{i}\theta}+c.c.=0,\\&p_2=\frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2u_2}{\partial t^2}-\alpha_0\frac{\partial^2u_2}{\partial z^2}+\beta_1u_2+\left\{\left[2\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\left(\alpha_1-\frac{\alpha_0}{\lambda_\theta}\right)k^2+2\beta_2\right]|U|^2+\beta_2h^2\right\}\\&\quad+\left[2\mathrm{i}\left(\alpha_0k-\frac{m\omega\lambda}{\lambda_\theta\lambda_z}\right)\frac{\partial U}{\partial\xi}-\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+2\beta_2+\alpha_1k^2\right)hU\right]\mathrm{e}^{\mathrm{i}\theta}\\&\quad+\left[\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\frac{k^2}{2}\left(3\alpha_1+\frac{\alpha_0}{\lambda_\theta}\right)+\beta_2\right]U^2\mathrm{e}^{2\mathrm{i}\theta}+c.c.,\end{aligned}\quad(29)$$

where $|U|^2=UU^*$, U^* is the complex conjugate of U.

The form of equations (29) suggests to seek the following type of solution:

$$u_2 = U_2^{(0)} + \left(\sum_{\ell=1}^{2} U_2^{(\ell)} \mathrm{e}^{\mathrm{i}\ell\theta} + c.c.\right),$$
$$w_2 = W_2^{(0)} + \left(\sum_{\ell=1}^{2} W_2^{(\ell)} \mathrm{e}^{\mathrm{i}\ell\theta} + c.c.\right). \quad (30)$$

Introducing (30) into $(29)_3$, we obtain

$$P_2^{(0)} = \beta_1 U_2^{(0)} + \left[2\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + \left(\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)k^2 + 2\beta_2\right]|U|^2 + \beta_2 h^2(\tau), \quad (31)$$

$$2\omega U_2^{(1)} - \lambda_\theta k W_2^{(1)} = 2\mathrm{i}\left(\frac{\omega}{k} - \lambda\right)\frac{\partial U}{\partial \xi} - 2\frac{\omega}{\lambda_\theta}h(\tau)U,$$
$$\omega W_2^{(1)} - k\left(-\frac{m\omega^2}{\lambda_\theta\lambda_z} + \alpha_0 k^2 + \beta_1\right)U_2^{(1)} =$$
$$\mathrm{i}\left(-2\frac{\lambda}{\lambda_\theta}\frac{\omega}{k} - 2m\omega\lambda\frac{k}{\lambda_\theta\lambda_z} - \frac{m\omega^2}{\lambda_\theta\lambda_z} + 3\alpha_0 k^2 + \beta_1\right)\frac{\partial U}{\partial \xi}$$
$$+ \left[4\frac{\omega^2}{\lambda_\theta^2 k}w_0 - k\left(\frac{m\omega^2}{\lambda_\theta\lambda_z} + 2\beta_2 + \alpha_1 k^2\right)\right]h(\tau)U, \quad (32)$$

$$2\omega U_2^{(2)} - \lambda_\theta k W_2^{(2)} = \frac{\omega}{\lambda_\theta}U^2,$$
$$\omega W_2^{(2)} - k\left(-\frac{4m\omega^2}{\lambda_\theta\lambda_z} + 4\alpha_0 k^2 + \beta_1\right)U_2^{(2)} =$$
$$\left[2\frac{\omega^2}{\lambda_\theta^2 k} + \frac{m\omega^2 k}{\lambda_\theta^2\lambda_z} + \frac{1}{2}\left(3\alpha_1 + \frac{\alpha_0}{\lambda_\theta}\right)k^3 + \beta_2 k\right]U^2. \quad (33)$$

Since, in this work, we will deal with first-order terms in the perturbation expansion and without loosing the generality of the problem, we may take $U_2^{(1)} = 0$. Thus, from (32) we obtain

$$\mathrm{i}\left[\frac{2\omega}{\lambda_\theta k}\left(\frac{\omega}{k} - \lambda\right) - \frac{2m\omega\lambda k}{\lambda_\theta\lambda_z} - \frac{m\omega^2}{\lambda_\theta\lambda_z} - \frac{2\lambda\omega}{\lambda_\theta k} + 3\alpha_0 k^2 + \beta_1\right]\frac{\partial U}{\partial \xi} + \left[\frac{4\omega^2}{\lambda_\theta^2 k}w_0 - k\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\beta_2 + \alpha_1 k^2 + \frac{2\omega^2}{\lambda_\theta^2 k^2}\right)\right]h(\tau)U = 0. \quad (34)$$

In order to have a non-zero solution for U, the following conditions must be satisfied:

$$\frac{2\omega}{\lambda_\theta k}\left(\frac{\omega}{k} - \lambda\right) - \frac{2m\omega\lambda k}{\lambda_\theta\lambda_z} - \frac{m\omega^2}{\lambda_\theta\lambda_z} - \frac{2\lambda\omega}{\lambda_\theta k} + 3\alpha_0 k^2 + \beta_1 = 0, \quad (35)$$

$$\frac{4\omega^2}{\lambda_\theta^2 k}w_0 - k\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\beta_2 + \alpha_1 k^2 + \frac{2\omega^2}{\lambda_\theta^2 k^2}\right) = 0. \quad (36)$$

From the solution of (35) we obtain the group velocity $\lambda = v_g$ as

$$\lambda = \frac{2\omega^2 + \lambda_\theta\alpha_0 k^4}{\omega k(2 + mk^2/\lambda_z)}. \quad (37)$$

The solution of (36) gives w_0 as

$$w_0 = \frac{k^2\lambda_\theta^2}{4\omega^2}\left[\left(2\beta_2 + \frac{\beta_1}{\lambda_\theta}\right) + \left(\alpha_1 + \frac{\alpha_0}{\lambda_\theta}\right)k^2\right]. \quad (38)$$

From the solution of $(32)_1$ we obtain $W_2^{(1)}$ as

$$W_2^{(1)} = \mathrm{i}\frac{2}{\lambda_\theta k}\left(\lambda - \frac{\omega}{k}\right)\frac{\partial U}{\partial \xi} + \frac{2\omega}{\lambda_\theta^2 k}h(\tau)U. \quad (39)$$

The solution of (33) yields

$$U_2^{(2)} = \Phi_0 U^2, \quad W_2^{(2)} = \frac{2}{\lambda_\theta}\frac{\omega}{k}U_2^{(2)} - \frac{\omega}{\lambda_\theta^2 k}U^2,$$
$$\Phi_0 = \frac{(\omega^2/\lambda_\theta + 3(\lambda_\theta\alpha_1 + \alpha_0)k^4/2 + (\lambda_\theta\beta_2 + \beta_1)k^2)}{3(\beta_1\lambda_\theta k^2 - 2\omega^2)}. \quad (40)$$

4.3. Solution of $O(\varepsilon^3)$ Equations

For our future purposes, we need the zeroth- and first-order equations in terms of the phasor θ. For this order of equations we express the variables as

$$u_3 = U_3^{(0)} + \left(\sum_{\ell=1}^{3} U_3^{(\ell)} \mathrm{e}^{\mathrm{i}\ell\theta} + c.c.\right),$$
$$w_3 = W_3^{(0)} + \left(\sum_{\ell=1}^{3} W_3^{(\ell)} \mathrm{e}^{\mathrm{i}\ell\theta} + c.c.\right), \quad (41)$$
$$p_3 = P_3^{(0)} + \left(\sum_{\ell=1}^{3} P_3^{(\ell)} \mathrm{e}^{\mathrm{i}\ell\theta} + c.c.\right).$$

Introducing the proposed solution (41) into equation (25), the zeroth- and the first-order equations may be given as

$$-2\lambda\frac{\partial U_2^{(0)}}{\partial \xi} + \lambda_\theta\frac{\partial W_2^{(0)}}{\partial \xi} + \frac{4}{\lambda_\theta}\frac{\omega}{k}\frac{\partial}{\partial \xi}|U|^2 + \frac{2\omega}{k}w_0\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = 0,$$

$$-\lambda\frac{\partial W_2^{(0)}}{\partial\xi}+\frac{\partial P_2^{(0)}}{\partial\xi}+\frac{4}{\lambda_\theta^2}\frac{\omega^2}{k^2}\frac{\partial}{\partial\xi}|U|^2-\beta_1\frac{h(\tau)}{\mathrm{d}\tau}=0, \tag{42}$$

$$\begin{aligned}&-2\omega U_3^{(1)}+k\lambda_\theta W_3^{(1)}+\mathrm{i}\lambda_\theta\frac{\partial W_2^{(1)}}{\partial\xi}+2\mathrm{i}\frac{\omega}{k}\frac{\partial U}{\partial\tau}+2\frac{\omega}{\lambda_\theta}U_2^{(2)}U^*+kW_2^{(2)}U^*+\left(kW_2^{(0)}+2\frac{\omega}{\lambda_\theta}U_2^{(0)}\right)U\\&+\mathrm{i}2\frac{\omega}{\lambda_\theta k}(w_0-1)h(\tau)\frac{\partial U}{\partial\xi}-kh(\tau)W_2^{(1)}=0,\\&-\omega W_3^{(1)}+kP_3^{(1)}-\mathrm{i}\lambda\frac{\partial W_2^{(1)}}{\partial\xi}+\mathrm{i}\frac{\partial P_2^{(1)}}{\partial\xi}+\mathrm{i}\frac{2}{\lambda_\theta}\frac{\omega^2}{k^2}\frac{\partial U}{\partial\tau}+2\frac{\omega}{\lambda_\theta}W_2^{(2)}U^*+2\frac{\omega}{\lambda_\theta}W_2^{(0)}U+\mathrm{i}\frac{4\omega^2}{\lambda_\theta^2k^2}w_0h(\tau)\frac{\partial U}{\partial\xi}\\&+2\frac{\omega}{\lambda_\theta}w_0h(\tau)W_2^{(1)}=0,\\&P_3^{(1)}=\left(\alpha_0k^2-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\beta_1\right)U_3^{(1)}+\left(\frac{m\lambda^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2U}{\partial\xi^2}+2\mathrm{i}\alpha_0k\frac{\partial U}{\partial\tau}\left[\frac{5m\omega^2}{\lambda_\theta^2\lambda_z}+\left(3\alpha_1-2\frac{\alpha_0}{\lambda_\theta}\right)k^2+2\beta_2\right]U_2^{(2)}U^*\\&+\left[\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\alpha_1k^2+2\beta_2\right]U_2^{(0)}U\left[-3\frac{m\omega^2}{\lambda_\theta^3\lambda_z}+(2\alpha_2-\frac{\alpha_1}{2\lambda_\theta}+\frac{\alpha_0}{2\lambda_\theta^2}\Big)k^2+\frac{3}{2}(\gamma_1-\alpha_0)k^4+3\beta_3\right]|U|^2U\\&-2\mathrm{i}\left(\frac{m\omega\lambda}{\lambda_\theta^2\lambda_z}+\alpha_1k\right)h(\tau)\frac{\partial U}{\partial\xi}+\left(-\frac{m\omega^2}{\lambda_\theta^3\lambda_z}+3\beta_3+\alpha_2k^2\right)h^2(\tau)U.\end{aligned} \tag{43}$$

The solution of the set (42) together with (31) yields the following result:

$$\begin{aligned}U_2^{(0)}&=\Phi_1|U|^2+\left(\frac{\lambda_\theta\beta_2}{2\lambda^2-\lambda_\theta\beta_1}\right)h^2(\tau)+\left[\frac{(2\omega\lambda/k\lambda_\theta)w_0-\beta_1\lambda_\theta}{2\lambda^2-\lambda_\theta\beta_1}\right]\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\xi,\\W_2^{(0)}&=2\frac{\lambda}{\lambda_\theta}U_2^{(0)}-\frac{4}{\lambda_\theta^2}\frac{\omega}{k}|U|^2-2\frac{\omega}{\lambda_\theta k}w_0\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\xi,\\\Phi_1&=\frac{\left[4\omega\lambda/\lambda_\theta k+2(\beta_1+\lambda_\theta\beta_2)+(\alpha_0+\lambda_\theta\alpha_1)k^2\right]}{2\lambda^2-\lambda_\theta\beta_1}.\end{aligned} \tag{44}$$

Finally, eliminating $U_3^{(1)}$, $W_3^{(1)}$ and $P_3^{(1)}$ between the equations (43) through the use of the dispersion relation (28), the following NLS equation with variable coefficients is obtained:

$$\mathrm{i}\frac{\partial U}{\partial\tau}+\mu_1\frac{\partial^2U}{\partial\xi^2}+\mu_2|U|^2U+\mathrm{i}\mu_3h(\tau)\frac{\partial U}{\partial\xi}+\left[\mu_4h^2(\tau)+\mu_5\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\xi\right]U=0, \tag{45}$$

where the coefficients μ_1, μ_2, μ_3, μ_4 and μ_5 are defined by

$$\begin{aligned}\mu_1&=\left[2\left(\alpha_0\lambda_\theta k^3+2\omega^2/k\right)\right]^{-1}\left[\lambda^2(2+mk^2/\lambda_z)-4\lambda\omega/k+2m\omega\lambda k/\lambda_z+2\omega^2/k^2-3\alpha_0\lambda_\theta k^2\right],\\\mu_2&=\left[2\left(\alpha_0\lambda_\theta k^3+2\omega^2/k\right)\right]^{-1}\Bigg\{\lambda_\theta k^2\Bigg[-3\frac{m\omega^2}{\lambda_\theta^3\lambda_z}+2\alpha_2k^2-\frac{\alpha_1}{2\lambda_\theta}k^2+\frac{\alpha_0}{2\lambda_\theta^2}k^2+3(\gamma_1-\alpha_0)k^4/2+3\beta_3\Bigg]\\&\quad-15\frac{\omega^2}{\lambda_\theta^2}+\left[6\frac{\omega\lambda k}{\lambda_\theta}+2\frac{\omega^2}{\lambda_\theta}+\lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\alpha_1k^2+2\beta_2\right)\right]\Phi_1+\left[8\frac{\omega^2}{\lambda_\theta}+\lambda_\theta k^2\left(5\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+3\alpha_1k^2-2\frac{\alpha_0}{\lambda_\theta}k^2+2\beta_2\right)\right]\Phi_0\Bigg\},\\\mu_3&=\left[2\left(\alpha_0\lambda_\theta k^3+2\omega^2/k\right)\right]^{-1}\left\{2\left(\frac{\omega^2}{\lambda_\theta k}+\frac{2\omega\lambda}{\lambda_\theta}\right)w_0-3\left(\alpha_0+\lambda_\theta\alpha_1\right)k^3-\left(\beta_1+2\lambda_\theta\beta_2\right)k\right\},\\\mu_4&=\left[2\left(\alpha_0\lambda_\theta k^3+2\omega^2/k\right)\right]^{-1}\Bigg\{-\frac{m\omega^2k^2}{\lambda_\theta^2\lambda_z}+3\beta_3\lambda_\theta k^2+\alpha_2\lambda_\theta k^4-2\frac{\omega^2}{\lambda_\theta^2}+4\frac{\omega^2}{\lambda_\theta^2}w_0+\frac{\lambda_\theta\beta_2}{2\lambda^2-\lambda_\theta\beta_1}\Bigg[2\frac{\omega^2}{\lambda_\theta}\\&\quad+6\frac{\omega\lambda k}{\lambda_\theta}+\frac{m\omega^2k^2}{\lambda_\theta\lambda_z}+\alpha_1\lambda_\theta k^4+2\lambda_\theta\beta_2k^2\Bigg]\Bigg\},\\\mu_5&=\left[2\left(\alpha_0\lambda_\theta k^3+2\omega^2/k\right)\right]^{-1}\Bigg\{-6\frac{\omega^2}{\lambda_\theta}w_0+\frac{2\frac{\omega}{k}\lambda w_0-\lambda_\theta\beta_1}{2\lambda^2-\lambda_\theta\beta_1}\Bigg[2\frac{\omega^2}{\lambda_\theta}+6\frac{\omega\lambda k}{\lambda_\theta}+\frac{m\omega^2k^2}{\lambda_\theta\lambda_z}\\&\quad+\lambda_\theta\alpha_1k^4+2\lambda_\theta\beta_2k^2\Bigg]\Bigg\}.\end{aligned} \tag{46}$$

It is seen that the coefficients depend both on ξ and τ. This variable coefficient NLS equation is the result of stenosis. If the stenosis function $h(\tau)$ is set equal to zero, we obtain the conventional NLS equation

$$\mathrm{i}\frac{\partial U}{\partial \tau}+\mu_1\frac{\partial^2 U}{\partial \xi^2}+\mu_2|U|^2U=0. \quad (47)$$

As a matter of fact, (45) can be reduced to the conventional NLS equation by introducing the new dependent variable

$$U(\xi,\tau)=V(\xi,\tau)\exp[\mathrm{i}\varphi(\xi,\tau)], \quad (48)$$

where the phase function $\varphi(\xi,\tau)$ is defined by

$$\varphi(\xi,\tau)=(\mu_4-\mu_1\mu_5^2-\mu_3\mu_5)\int_0^\tau h^2(s)\mathrm{d}s + \mu_5 h(\tau)\xi. \quad (49)$$

Noting the differential relations

$$\frac{\partial U}{\partial \tau}=\left\{\frac{\partial V}{\partial \tau}+\mathrm{i}\Big[(\mu_4-\mu_1\mu_5^2-\mu_3\mu_5)h^2(\tau) + \mu_5\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\xi\Big]V\right\}\exp[\mathrm{i}\varphi(\xi,\tau)],$$
$$\frac{\partial U}{\partial \xi}=\left[\frac{\partial V}{\partial \xi}+\mathrm{i}\mu_5 h(\tau)V\right]\exp[i\varphi(\xi,\tau)], \quad (50)$$
$$\frac{\partial^2 U}{\partial \xi^2}=\left[\frac{\partial^2 V}{\partial \xi^2}+2\mathrm{i}\mu_5 h(\tau)\frac{\partial V}{\partial \xi} - \mu_5^2 h^2(\tau)V\right]\exp[\mathrm{i}\varphi(\xi,\tau)],$$

and introducing (50) into (45), we have

$$\mathrm{i}\frac{\partial V}{\partial \tau}+\mu_1\frac{\partial^2 V}{\partial \xi^2}+\mu_2|V|^2V + \mathrm{i}(\mu_3+2\mu_1\mu_5)h(\tau)\frac{\partial V}{\partial \xi}=0. \quad (51)$$

Now, introducing the coordinate transformations

$$\tau'=\tau,\quad \xi'=\xi-(\mu_3+2\mu_1\mu_5)\int_0^\tau h(s)\mathrm{d}s \quad (52)$$

and inserting them into (51), we obtain the conventional NLS equation

$$\mathrm{i}\frac{\partial V}{\partial \tau'}+\mu_1\frac{\partial^2 V}{\partial \xi'^2}+\mu_2|V|^2V=0. \quad (53)$$

5. Progressive Wave Solution

In this subsection we will propose the progressive wave solution to (53) of the form

$$V(\xi',\tau')=F(\zeta)\exp[\mathrm{i}(K\xi'-\Omega\tau')],\quad \zeta=\beta(\xi'-2\mu_1 K\tau'), \quad (54)$$

where Ω, K and β are some constants and $F(\zeta)$ is a real-valued unknown function to be determined from the solution. Introducing (54) into (53) we have

$$\mu_1\beta^2F''+(\Omega-\mu_1K^2)F+\mu_2F^3=0, \quad (55)$$

where the prime denotes the differentiation of the corresponding quantity with respect to ζ.

As is well known, this equation admits two types of solution depending on the sign of the product $\mu_1\mu_2$. If $\mu_1\mu_2>0$, the solution may be given by

$$F(\zeta)=a\,\mathrm{sech}\,\zeta, \quad (56)$$

with

$$\beta=\left(\frac{\mu_2}{2\mu_1}\right)^{1/2}a,\quad \Omega=\mu_1K^2-\frac{\mu_2}{2}a^2. \quad (57)$$

If $\mu_1\mu_2<0$, the solution may be given by

$$F(\zeta)=a\,\tanh\zeta, \quad (58)$$

with

$$\beta=\left(-\frac{\mu_2}{2\mu_1}\right)^{1/2}a,\quad \Omega=\mu_1K^2-\mu_2a^2. \quad (59)$$

Thus, the solution of U in terms of the variables (ξ,τ) may be given by

$$U=a\,\mathrm{sech}\,\zeta\exp\{\mathrm{i}[\psi_2(\tau)\xi-\psi_1(\tau)]\}, \quad (60)$$

where the functions ψ_1, ψ_2 and ζ are defined by

$$\psi_1(\tau)=\Omega\tau+K(\mu_3+2\mu_1\mu_5)\int_0^\tau h(s)\mathrm{d}s - (\mu_4-\mu_1\mu_5^2-\mu_3\mu_5)\int_0^\tau h^2(s)\mathrm{d}s,$$
$$\psi_2(\tau)=K+\mu_5h(\tau),$$
$$\zeta=\left(\frac{\mu_2}{2\mu_1}\right)^{1/2}a\Big[\xi-(\mu_3+2\mu_1\mu_5)\int_0^\tau h(s)\mathrm{d}s - 2\mu_1K\tau\Big]. \quad (61)$$

Here the solution given in (60) is quite different from the solution of the conventional NLS equation. The trajectories both for the harmonic and enveloping waves are not straight lines anymore; they are rather some curves in the (ξ, τ) plane. This is the result of the stenosis in the elastic tube. If one sets the stenosis function $h(\tau)$ equal to zero, the solutions (60) and (61) reduce to the solution of the conventional NLS equation.

In the present derivation τ is the space and ξ is the time variable. Therefore, the speed of the wave may be defined by $\mathrm{d}\tau/\mathrm{d}\xi$. Applying this definition to the enveloping and harmonic waves we obtain the speeds of propagation as

$$v_\mathrm{e} = \frac{1}{2\mu_1 K + (\mu_3 + 2\mu_1\mu_5)h(\tau)}, \tag{62}$$

$$\begin{aligned} v_\mathrm{h} = {} & \left[K + \mu_5 h(\tau)\right]\left[\Omega + K(\mu_3 + 2\mu_1\mu_5)h(\tau)\right. \\ & \left. - (\mu_4 - \mu_1\mu_5^2 - \mu_3\mu_5)h^2(\tau) - \mu_5 h'(\tau)\xi\right]^{-1}. \end{aligned} \tag{63}$$

As is seen from (62) and (63) the speeds of both waves are variable. This is the result of stenosis in the tube.

In general, due to the complex structure of the coefficients $\mu_1, \ldots, \mu_5$, it is quite involved to study the variation of the wave speeds with the parameters ξ and τ. For that purpose we will investigate the long wave length or short wave number limit.

In this limiting case the coefficients $\mu_1, \ldots, \mu_5$ take the following form:

$$\begin{aligned} \mu_1 &= -\frac{3}{4}\delta k, \\ \mu_2 &= \frac{(4/\lambda_\theta + 2\beta_2/\beta_1)(5/\lambda_\theta + 2\beta_2/\beta_1)}{3\delta}\frac{1}{k}, \\ \mu_3 &= \frac{1}{4\lambda_\theta} + \frac{\beta_2}{2\beta_1}, \\ \mu_4 &= \frac{2(2/\lambda_\theta + \beta_2/\beta_1)\beta_2/\beta_1}{3\delta}\frac{1}{k}, \\ \mu_5 &= -\frac{(1/\lambda_\theta + 2\beta_2/\beta_1)(2/\lambda_\theta + \beta_2/\beta_1)}{3\delta}\frac{1}{k}, \\ \delta &= \frac{2\alpha_0}{\beta_1} - \frac{m}{\lambda_z}. \end{aligned} \tag{64}$$

For the numerical evaluation of the wave speeds we have to know the constitutive relation of the tube material. In the present work, we will employ the stress-strain relations proposed by Demiray [16] for soft biological tissues:

$$\Sigma = \frac{1}{2\alpha}\left\{\exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right] - 1\right\}, \tag{65}$$

where α is a material constant; its numerical value for canine abdominal artery was found to be $\alpha = 1.948$ [17]. Introducing (65) into (18), the explicit expressions of the coefficients α_0, α_1, β_1, β_2 may be given as

$$\begin{aligned} \alpha_0 &= \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)G(\lambda_\theta, \lambda_z), \\ \alpha_1 &= \left[\frac{1}{\lambda_\theta^4\lambda_z^3} + \alpha\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)\left(1 - \frac{1}{\lambda_\theta^4\lambda_z^2}\right)\right] \\ &\quad \cdot G(\lambda_\theta, \lambda_z), \\ \beta_1 &= \left[\frac{4}{\lambda_\theta^5\lambda_z^3} + 2\frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right]G(\lambda_\theta, \lambda_z), \\ \beta_2 &= \left[-\frac{10}{\lambda_\theta^6\lambda_z^3} + \frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1 + \frac{11}{\lambda_\theta^4\lambda_z^2}\right)\right. \\ &\quad \left. + 2\frac{\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\right]G(\lambda_\theta, \lambda_z), \end{aligned} \tag{66}$$

where the function G is defined by

$$G(\lambda_\theta, \lambda_\theta) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right]. \tag{67}$$

Using this numerical value of the material coefficient α, the coefficients $\mu_1, \ldots, \mu_5$ are calculated for the initial deformation $\lambda_\theta = \lambda_z = 1.6$ and the wave number $k = 0.5$; the result is found to be $\mu_1 = -0.176$, $\mu_2 = 75.76$, $\mu_3 = 1.819$, $\mu_4 = 43.25$, $\mu_5 = -47.31$. Furthermore, choosing the wave number K and the wave amplitude a as unity and using these values of the coefficients $\mu_1, \ldots, \mu_5$ in the expressions of the wave speeds given in (62) and (63) we obtain

$$\begin{aligned} v_\mathrm{e} &= \frac{1}{-0.532 + 1.847\,\mathrm{sech}\,\tau}, \\ v_\mathrm{h} &= \left(1 - 4.731\,\mathrm{sech}\,\tau\right)\left(-38.06 + 1.847\,\mathrm{sech}\,\tau\right. \\ &\quad \left. - 5.232\,\mathrm{sech}^2\tau - 4.731\xi\,\mathrm{sech}\,\tau\tanh\tau\right)^{-1}. \end{aligned} \tag{68}$$

Here we have chosen the function $h(\tau)$ as $h(\tau) = 0.1\,\mathrm{sech}\,\tau$ (axially symmetric stenosis). Using the expressions of the wave speeds given in (68), the variation of them with the distance parameter τ is studied numerically and the results are depicted in Figs. 1 and 2. Figure 1 shows that the speed of the enveloping wave increases with increasing axial coordinates. On the other hand, Fig. 2 reveals that the speed of harmonic waves decreases with increasing axial coordinates.

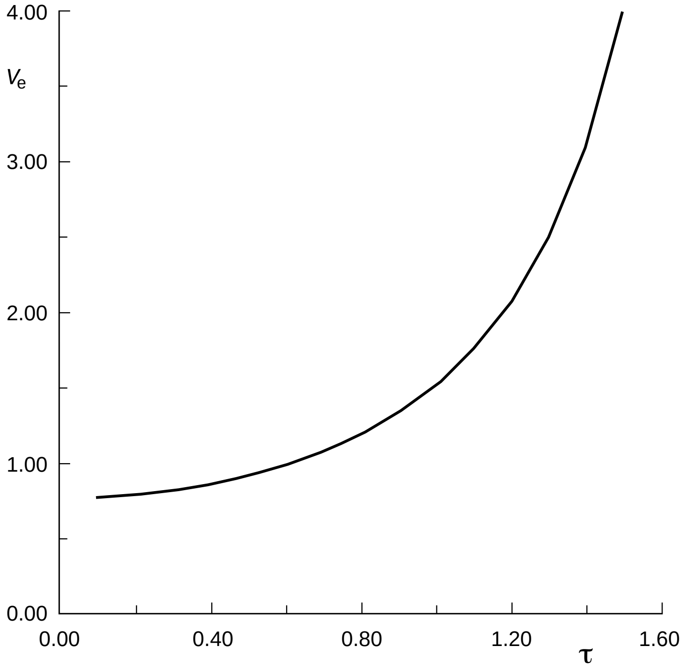

Fig. 1. Variation of enveloping wave speed with the distance parameter τ.

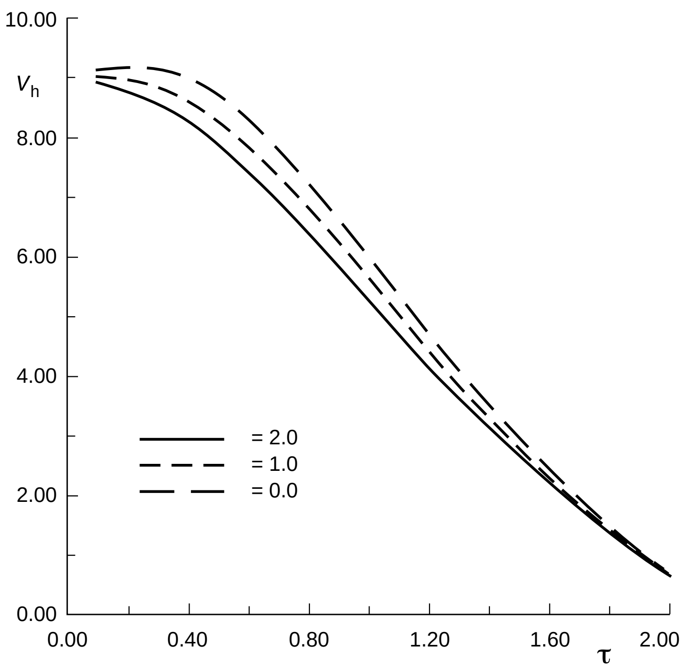

Fig. 2. Variation of harmonic wave speed with the distance parameter τ and time parameter ξ.

6. Results and Discussion

In the present study, by treating the arteries as a prestressed thin elastic tube with a symmetric stenosis and the blood as an incompressible Newtonian fluid with negligible viscosity, we have studied the propagation of weakly nonlinear waves in such a composite medium by use of the reductive perturbation method. The governing evolution equation was obtained as the variable coefficient nonlinear Schrödinger (NLS) equation. It was observed that by setting the stenosis function equal to zero, the evolution equation reduces to the conventional NLS equation. After introducing a new dependent function and a set of new independent coordinates, we reduced the variable coefficient NLS equation to the conventional NLS equation in the new variables. By seeking a progressive wave type of solution to this evolution equation we have observed, that the wave trajectories are not straight lines anymore; they are rather some curves in the (ξ, τ) plane. It was further noticed that the wave speeds for both enveloping and harmonic waves are variable in the axial coordinate, and the speed of the enveloping wave increases with increasing axial distance, whereas the speed of the harmonic wave decreases with increasing axial coordinates. The numerical calculations indicated that the speed of the harmonic wave decreases with increasing time parameter, but the sensitivity of the enveloping wave speed to this parameter is quite weak.

Acknowledgements

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